GONALITY OF CURVES ON FUNDAMENTAL LOCI OF FIRST ORDER CONGRUENCES (APPENDIX TO ARTICLE OF EIN, LAZARSFELD AND ULLERY)

FRANCESCO BASTIANELLI AND PIETRO DE POI

Abstract. This short paper is an appendix to [6]. We prove an existence result for families of curves having low gonality, and lying on fundamental loci of first order congruences of lines in $\mathbf{P}^{n+1}$. As an application, we follow the ideas of the main paper, and we present a slight refinement of a theorem included in it. In particular, we show that given a very general hypersurface $X \subset \mathbf{P}^{n+1}$ of degree $d \geq 3n-2 \geq 7$, and a dominant rational map $f\colon X \dashrightarrow \mathbf{P}^n$, then $\deg(f) \geq d-1$, and equality holds if and only if f is the projection from a point of X.

1. Results

Congruences of lines in the complex projective space $\mathbf{P}^{n+1}$ are very classical geometric objects, which are systematically studied by describing the geometry of the associated fundamental loci (see e.g. [5] for an extended survey on the subject). In this appendix to [6], we prove the existence of families of curves having low gonality, and covering fundamental loci of first order congruences of lines in $\mathbf{P}^{n+1}$. Actually, our initial motivation for dealing with this issue was to investigate the degree of irrationality of smooth hypersurfaces $X \subset \mathbf{P}^{n+1}$. We indeed apply our result to the latter problem and, combining the approaches of [6] and [1], we present a slight refinement of [6, Theorem C]. Namely, denoting by $\operatorname{irr}(X) := \min\{\delta > 0 \,|\, \exists \text{ a rational mapping } X \dashrightarrow \mathbf{P}^n \text{ of degree } \delta\}$ the *degree of irrationality* of X, we deduce the following.

Theorem 1.1. *Let $X \subset \mathbf{P}^{n+1}$ be a very general smooth hypersurface of degree $d \geq 3n-2$, with $n \geq 3$. Then $\operatorname{irr}(X) = d-1$. Furthermore, any rational mapping $f\colon X \dashrightarrow \mathbf{P}^n$ of degree $d-1$ is given by projection from a point of X.*

It is worth noticing that the assertion is covered by [6, Theorem C] as long as $d \geq 3n$, so Theorem 1.1 simply relaxes such a hypothesis to $d \geq 3n-2$ when $n \geq 3$.

A *first order congruence of lines in* $\mathbf{P}^{n+1}$ is a family $W \xrightarrow{\pi} B$ of lines parameterized over a desingularization of a n-dimensional subvariety $B_0 \subset \mathbf{G}(\mathbf{P}^1, \mathbf{P}^{n+1})$ of the Grassmannian of lines in $\mathbf{P}^{n+1}$, with the property that there is only one line of the family passing through a general point of $\mathbf{P}^{n+1}$. The *fundamental locus* $\Phi \subset \mathbf{P}^{n+1}$ of the congruence is the locus of points lying on infinitely many lines of the family.

In order to state our result, we consider an irreducible (possibly non-reduced) component $Z \subset \Phi$, and we assume $\dim Z > 0$. According to [6], we recall that the *covering gonality* of Z_{red}, denoted by $\operatorname{cov.gon}(Z_{\mathrm{red}})$, is the least integer $c > 0$ for which there exist a smooth family $\mathcal{C} \xrightarrow{\phi} T$ of irreducible curves and a dominant morphism $\varphi\colon \mathcal{C} \longrightarrow Z_{\mathrm{red}}$, such that the general fibre $C_t := \phi^{-1}(t)$ is a smooth

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c-gonal curve mapping birationally onto its image under φ. Finally, we associate to Z a non-negative integer $m(Z)$ as follows. Let $\ell_b \subset \mathbb{P}^{n+1}$ be a general line of the congruence. We set $m(Z) = 0$ if and only if the intersection $\ell_b \cap Z$ is empty. If instead ℓ_b meets Z, then the intersection $\ell_b \cap Z$ outside other irreducible components of Φ consists of a 0-dimensional scheme $P_1 \cup \cdots \cup P_k$, where each P_j is a fat point of length h_j. For any h_j, we denote by m_j the cardinality of the set $\{1 \leq i \leq k \,|h_i = h_j\}$, and we define $m(Z) := \min\{m_j \,|1 \leq j \leq k\}$. Therefore, we prove the following result, which relates the covering gonality of Z_{red} to the integer $m(Z)$.

Proposition 1.2. *Let $Z \subset \Phi$ be an irreducible component of the fundamental locus of a first order congruence of lines in $\mathbf{P}^{n+1}$, with $\dim Z > 0$. Then the covering gonality of Z_{red} satisfies*

(i) $\operatorname{cov.gon}(Z_{\mathrm{red}}) = 1$ *when* $0 \leq m(Z) \leq 2$*;*
(ii) $\operatorname{cov.gon}(Z_{\mathrm{red}}) \leq m(Z) - 1$ *for any* $m(Z) \geq 3$.

Moreover, Z_{red} is covered by lines as $m(Z) = 0$, whereas Z_{red} is unirational when $m(Z) = 1$.

In particular, since the intersection between Φ and a general line of the congruence consists of at most n distinct points, we deduce that $\operatorname{cov.gon}(Z_{\mathrm{red}}) \leq n - 1$ for any such a component $Z \subset \Phi$.

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2. Proofs

Let $W \xrightarrow{\pi} B$ be a first order congruence of lines in $\mathbf{P}^{n+1}$, where $B \longrightarrow B_0$ is a desingularization of a n-dimensional subvariety $B_0 \subset \mathbf{G}(\mathbf{P}^1, \mathbf{P}^{n+1})$. Then the congruence is the $\mathbf{P}^1$-bundle obtained as the pullback to B of the incidence correspondence $W_0 := \left\{(b, y) \in B_0 \times \mathbf{P}^{n+1} \,|y \in \ell_b\right\}$, where $\ell_b \in \mathbf{P}^{n+1}$ denotes the line parameterized by $b \in B_0$. In particular, we have a diagram

$$\begin{array}{ccc} E \subset W & \xrightarrow{\mu} & \mathbf{P}^{n+1} \supset \Phi \\ {\scriptstyle\pi}\big\downarrow & & \\ B & & \end{array} \tag{2.1}$$

where μ is the natural birational morphism mapping each fibre $L_b := \pi^{-1}(b)$ to the line $\ell_b \in \mathbf{P}^{n+1}$, the ramification divisor $E = \{w \in W \,|\mu \text{ is not a local isomorphism at } w\}$ is the *exceptional locus* of μ, and $\Phi \subset \mathbf{P}^{n+1}$ is the fundamental locus of the congruence. Moreover, Φ has dimension $\dim \Phi \leq n - 1$, and it is the scheme-theoretic image of E under μ (see e.g. [5, Theorem 2.8]). The proof of Proposition 1.2 relies indeed on the ruledness of exceptional loci of birational mappings.

Proof of Proposition 1.2. Let $W \xrightarrow{\pi} B$ be a first order congruence of lines in $\mathbf{P}^{n+1}$, and let Z be a positive dimensional component of the corresponding fundamental locus Φ. If $m(Z) = 0$, then the general line of the congruence does not intersect Z. Thus [4, Proposition 2.12] implies that Z is a so-called *parasitical scheme* of the congruence, which is covered by lines (cf. [4, Section 2.3]).

So, let $m(Z) > 0$, and consider the 0-dimensional scheme $P_1 \cup \cdots \cup P_k$ defined on Z by a general line of the congruence, where each P_j is a fat point of length h_j. Without loss of generality, we may assume that $m(Z)$ is computed by $h_1 = \cdots = h_{m(Z)}$, so that $h_j \neq h_1$ for any $m(z) + 1 \leq j \leq k$. We point out that for general $z \in Z_{\mathrm{red}}$, there exists some line $\ell_b \subset \mathbf{P}^{n+1}$ passing through z, such that the fat point supported at z has length h_1 (otherwise, the points of length h_1 cut out on Z by the

lines of the congruence would describe a different irreducible component $Z' \subset \Phi$, with $Z'_{\text{red}} \subsetneq Z_{\text{red}}$). Since B is n-dimensional, those lines form a (possibly reducible) subfamily of $W \xrightarrow{\pi} B$ of dimension equal to $n - \dim Z = \dim \mu^{-1}(z)$. If in addition $m(Z) \geq 2$, assuming that P_1 is supported at z, and varying the line through z in such a family, the points $P_2, \dots, P_{m(Z)}$ describe a subvariety of Z. By arguing as above, we then deduce that, as $z \subset Z_{\text{red}}$ varies, these subvarieties cover Z.

If $m(Z) = 1$, we consider a general linear subspace $H \subset \mathbf{P}^{n+1}$ of dimension $\dim H = \dim Z$. As the congruence has order one, there is a unique line through a general point $y \in H$, and such a line cuts out on Z a unique point of length h_1. Hence we can define a dominant rational map $H \dashrightarrow Z_{\text{red}}$, which sends $y \in H$ to the point $z \in Z_{\text{red}}$ supporting the fat point of length h_1. Thus Z_{red} is unirational.

Then we set $m(Z) \geq 2$, and we prove that $\operatorname{cov.gon}(Z_{\text{red}}) \leq m(Z) - 1$. This is equivalent to show that given a general point $z' \in Z$, there exists an irreducible curve C' through z' having gonality $\operatorname{gon}(C') \leq m(Z) - 1$, where $\operatorname{gon}(C')$ is defined as the gonality of its normalization (cf. [6]). Since $\mu \colon W \longrightarrow \mathbf{P}^{n+1}$ is a birational morphism between smooth varieties, [8, Lemma 1.6.2, p. 289] assures that every irreducible component E' of the exceptional locus $E \subset W$ of μ is ruled. So we consider a general point $z \in Z_{\text{red}}$, and its fibre $\mu^{-1}(z)$. Let R be a curve of the ruling of some $E' \subset W$ such that $R \cap \mu^{-1}(z)$ is non-empty. If R is not contracted by μ, the assertion follows as Z_{red} would be covered by rational curves.

Thus we assume that μ contracts every such a rational curve R, so that any irreducible component of $\mu^{-1}(z) \subset E$ is ruled. Clearly, if $L_b := \pi^{-1}(b)$ is general among fibres intersecting $\mu^{-1}(z)$, the corresponding line $\ell_b := \mu(L_b) \subset \mathbf{P}^{n+1}$ passes through z, and the intersection $L_b \cap \mu^{-1}(z)$ consists of a single point. As we observed above, the lines of the congruence intersecting Z at a fat point of lenght h_1 supported on z form a family of dimension $n - \dim Z = \dim \mu^{-1}(z)$. Therefore there exists an irreducible component M_z of $\mu^{-1}(z)$ such that if $L_b \cap \mu^{-1}(z)$ is non-empty, then ℓ_b belongs to such a family. So, we consider a general rational curve R of the ruling of M_z, and its image $\pi(R) \subset B$. We note that for general $b \in \pi(R)$, L_b is the only fibre passing through the point $R \cap L_b$. Moreover, denoting by $E^\circ \subset E$ the pre-image under μ of the open set of points of Z_{red} which do not lie on other components of Φ, we have that L_b meets E° at k distinct points, and these points are mapped by μ to the 0-dimensional scheme $P_1 \cup \cdots \cup P_k \subset Z$ cut out by ℓ_b. Then we define a (possibly reducible) curve $D \subset E$ as the Zariski closure of the set

$$\{w \in (E^\circ - R) \,|\, w \in L_b \text{ for some } b \in \pi(R), \text{ and } \mu(w) \in \ell_b \text{ has length } h_1\},$$

i.e. D is described by the $m(Z) - 1$ points of $L_b \cap E^\circ$ mapping to the fat points $P_2, \dots, P_{m(Z)} \in Z$ of length h_1, where we set $P_1 = \mu(L_b \cap R)$. Thus we can define a map $D \longrightarrow \pi(R)$ of degree $m(Z) - 1$, by sending $w \in L_b$ to $b \in \pi(R)$. In particular, any irreducible component $D' \subset D$ is not contracted by μ, and satisfies $\operatorname{gon}(D') \leq m(Z) - 1$. It follows that $C' := (\mu(D'))_{\text{red}} \subset Z_{\text{red}}$ is a curve satisfying $\operatorname{gon}(C') \leq m(Z) - 1$, as well. Therefore, as we vary the curve R in the ruling of M_z and $z \in Z_{\text{red}}$, we obtain infinitely many curves C' covering Z_{red}, with gonality bounded by $m(Z) - 1$. □

The proof of Theorem 1.1 depends on [6, Proposition 2.5], asserting that a very general hypersurface $X \subset \mathbf{P}^{n+1}$ of degree $d \geq 2n$ does not contain curves having gonality smaller than $d - 2n + 1$.

Proof of Theorem 1.1. Let $f \colon X \dashrightarrow \mathbf{P}^n$ be a rational mapping of degree $\deg(f) \leq d - 1$. Aiming for a contradiction, we assume that f is not the projection from a point of X. Then [1, Theorem 4.3] assures that f defines a first order congruence $W \xrightarrow{\pi} B$ of lines in $\mathbf{P}^{n+1}$, whose fundamental

locus Φ possesses an irreducible component $Z \subset \Phi$ such that $Z_{\mathrm{red}} \subset X$. In particular, $\dim Z > 0$ as f is not the projection from a point.

Thanks to the classification theorem [3, Theorem 5], we deduce that the intersection between Φ and the general line of the congruence consists of at most n distinct points, and hence $m(Z) \leq n$. Furthermore, the same result assure that $m(Z) = n$ if and only if $Z = \Phi$ is an integral $(n-1)$-fold, and the congruence consists of n-secant lines to Z. However, if $W \xrightarrow{\pi} B$ were such a congruence, then Z_{red} would not be a complete intersection in $\mathbf{P}^{n+1}$ by [4, Corollary 2.19], so it could not be contained in X by Lefschetz Theorem (see e.g. [7, Chapter IV]). Thus $m(Z) \leq n-1$, and Proposition 1.2 gives that the covering gonality of Z_{red} satisfies $\operatorname{cov.gon}(Z_{\mathrm{red}}) \leq n-2$.

In particular, the hypersurface $X \subset \mathbf{P}^{n+1}$ contains curves of gonality smaller that $n-1$. Hence [6, Proposition 2.6] implies that $d \leq 3n-3$, a contradiction. Thus $f\colon X \dashrightarrow \mathbf{P}^n$ is the projection from a point of X, with $\deg(f) = d-1$. □

Remark 2.1. We would like to note that when $n = 4$, Theorem 1.1 can be proved independently from the results of [6]. Arguing as in the latter proof, and working out the combinatorics given by [3, Theorem 5], Proposition 1.2 assures that each possible $Z \subset \Phi \subset \mathbf{P}^5$ is covered by rational curves, with two exceptions. In both of them Z is a reduced threefold, and the corresponding congruences are given either by 4-secant lines to Z, or by 3-secant lines to Z meeting another reduced threefold Z'. In the latter case, a generalization of [5, Theorem 7.1] and [1, Proposition 3.8] shows that Z is still covered by rational curves. Thus the non existence of rational curves on $X \subset \mathbf{P}^5$ (see e.g. [2]) and [4, Corollary 2.19] lead to the assertion.

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Dipartimento di Matematica e Fisica, Università degli Studi Roma Tre, Largo San Leonardo Murialdo 1, 00146 Roma - Italy

E-mail address: bastiane@mat.uniroma3.it

Dipartimento di Matematica e Informatica, Università degli Studi di Udine, via delle Scienze 206, 33100 Udine - Italy

E-mail address: pietro.depoi@uniud.it